SAFETY OF ARCHES - A PROBABILISTIC APPROACH

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ABSTRACT

Nowadays, powerful methods are available for the calculation of structural safety values. These permit to calculate the global failure probability of complex structures, relying on deterministic techniques able to determine the stability state for a prescribed set of parameters. The safety of arches is a typical example of new fields for these methods. Traditionally their stability is assessed using a deterministic approach, resulting in safety factors. To relate these safety factors to an absolute safety, requires a great deal of engineering judgement. In case of an existing arch, this judgement depends on a variety of uncertainties such as on the accuracy of the geometry measurements, the uncertainty on the material properties, support conditions and the constructional history of the structure. The proposed method calculates an absolute value for the global probability of failure at level III on the basis of the quantified uncertainties. Several techniques are available for the calculation of the global probability of failure at level III such as Monte Carlo, first order reliability method or second order reliability method (FORM/SORM) in combination with a system analysis. The recently developed directional adaptive response surface sampling method (DARS) meets both requirements of accuracy and efficiency. This method will be used to calculate the global probability of failure of arches in combination with the limit analysis code Calipous.

Key words: reliability analysis, failure probability, arch, structural safety, DARS, assessment

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INTRODUCTION

Reliability Analysis of Structural Systems

Assessing the safety of structural masonry involves many uncertainties. This paper presents a method that permits to build in these uncertainties as good as possible. The Joint Committee of Structural Safety defines three levels at which the structural safety can be assessed (JCCS 1982), Table 1. These were adopted in the European Standard Eurocode 1 (EC1 1994). Level III methods are the most accurate. Level I and level II methods are simplified approaches introduced for computational reasons. Ideally, they should be calibrated using a level III method.

Table 1. Different levels for the calculation of structural safety values

Level	Definition
Level III	Level III methods such as MC sampling and Numerical Integration are considered most accurate. They compute the exact probability of failure of the whole structural system, or structural elements, using the exact probability density function of all random variables.
Level II	Level II methods such as FORM and SORM compute the probability of failure by means of an idealization of the limit state function where the probability density functions of all random variables are approximated by equivalent normal distribution functions.
Level I	Level I methods verify whether or not the reliability of the structure is sufficient instead of computing the probability of failure explicitly. In practice this is often carried out by means of partial safety factors.

Even for level III methods it has to be stressed that some assumptions are made:
(1) all the variables and parameters driving the structures' behavior are known,
(2) the deterministic model relating them gives an exact estimate of the behavior,
(3) the parameters' exact distribution is known (as well as the resulting joint probability),
(4) the integration of the joint probability on the safe domain is exact.

For the assumptions (1) and (2), it is noticed that these or not inherent to the probabilistic approach itself. These assumptions are made in a deterministic stability state calculation as well. Assumption (3) is a best estimate of the uncertainty on the parameters. For several parameters this will be a good estimate of the real distribution, in others the estimate will be worse. But in every case, it remains the common used scientific approach for quantifying the uncertainties. Finally, the last assumption (4) depends on the method used. In case of a Monte Carlo or Directional Sampling analysis, an infinite amount of simulations is required to obtain the exact result. If the number of simulations is limited, an estimate of the exact value will be obtained. A number of simulations of the order of $3/p_f$ is required to achieve an acceptable level of accuracy (Melchers 1999). The low probabilities that can be expected in normal practice lead to a high number of evaluations. In case of the safety of arches, this would mean that the limit state function should be evaluated for a high number of different sets of input parameters, each of them requiring a call to the external program Calipous. Calipous is the limit analysis code developed by Smars (Smars 2000). Importance sampling is used to direct the sampling into the interesting regions and to increase the accuracy within a limited number of samples.

Using FORM, the limit state function will be linearized in the design point. In case of SORM, a second order approximation is used in the design point. Again, both will result in an estimate of the exact value. Therefor, FORM/SORM analyses require the first/second order derivatives of the limit state function in the design point. In the case of arches, they are not known analytically as the limit state function is implicit. Numerical estimations would imply extra limit state function evaluations, resulting in the same disadvantage as the MC method. An additional disadvantage comes from the existence of several potential failure modes for arches. Their correlation demands a system analysis to assess the probability of failure of the whole system and not only of a particular failure mode.

Recently a combined method has been developed: Directional Adaptive Response surface Sampling (DARS) (Waarts 2000). It is used when the limit state function is only known implicitly, but can also be interesting for computational reasons. It combines direct and indirect evaluations of the limit state function to obtain an optimal use of the response surface and to limit the amount of direct limit state function evaluations. In the indirect evaluations, an estimated response surface is used instead of the original limit state function. This method gains ground when each limit state function evaluation requires a time-consuming calculation, such as a non-linear finite element analysis.

Table 2: Overview of reliability methods for a level III reliability analysis

Level III methods	Direct/Indirect (D/ID)
Numerical integration (NI)	D
Monte Carlo (MC)	D
Importance Sampling Monte Carlo (ISMC)	D
Crude Directional Sampling (DS)	D
First Order and Second Order Reliability Method (Form and Sorm) (level II method) in combination with a system analysis	D
Form/Sorm with an adaptive Response Surface (level II method) in combination with a system analysis	D-ID
Directional Adaptive Response surface Sampling (DARS)	D-ID

The DARS method meets the disadvantages of the other methods to a certain extend. The limit state function evaluations are used to estimate the response of the system. This permits to reduce the number of evaluations for a given accuracy on the estimate of the failure probability. On the other hand, the probability of failure of the whole structural

system is calculated, as there is no preference for a certain failure mode. In the following, the DARS method will be applied to compute the safety of masonry arches.
The above mentioned methods that are available to assess the probability of failure on level III, are listed in Table 2. None is optimal for every particular case. The choice depends essentially on the complexity of the problem. The methods are subdivided into two categories: direct (D) and indirect (ID) as they call for a direct or indirect limit state function evaluation.

DARS - DIRECTIONAL ADAPTIVE RESPONSE SURFACE SAMPLING

The response surface (RS) method is used when the response is only available from experiments, complex finite element computations or a limit analysis using the Calipous code as in the case of masonry arches. An analytical limit state function replaces the real response function. The main idea is that the real response being a complex function of input variables, is approximated by a simple function of the input variables. In many cases, a polynomial of low order is sufficient to describe the response of the structure. Usually a polynomial of first or second order is used in practice including a constant term, linear terms, quadratic terms and cross-terms, Eq.1:

$$g_{RS} = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j \qquad (1)$$

In case this would not be sufficient to describe the response accurately, higher order polynomials or other analytical relationships should be looked for (Montgomery 1997).

The standard reliability procedure using a response surface (indirect method) can be outlined as follows:
Step 1. Selection and definition of the most important random variables. The number of variables is limited to a finite number *n* leaving out possibly an infinite set of parameters that in the model idealization process have been judged to be of secondary or negligible importance for the problem at hand (Ditlevsen 1982). Remark that this requires a certain amount of engineering judgement. The same of course holds for a deterministic analysis.
Step 2. Design of an optimal scheme of points at which the outcome is calculated, using a deterministic analysis (a limit analysis in this case). Each variable can be changed individually or more efficient combinations can be used according to an Optimal Design of Experiments (ODE) (Montgomery, 1997). For each set of variables a call to the external program is made and the outcome is calculated.
Step 3. Construction of a response surface through these response data using a least square analysis.
Step 4. Performing a reliability analysis on the response surface instead of the real problem. The type of reliability method (MC, DS, FORM/SORM or other) is of little importance since the time consuming evaluations using an external program are replaced by the analytical expression.

This method has some major disadvantages with respect to the required number of limit state function evaluation (LSFE) and accuracy:

•The amount of samples (=LSFE) equals $2n+1$ in case the random variables are individually changed, using $\mu_i \pm f.\sigma_i$, . This method does not at all guarantee a reliable response surface. For an optimal design, $2^n + 2n$ samples (=LSFE) are required in case of a so-called "Central Composite Design". For a high number of random variables (n) this leads to an enormous number of direct limit state function evaluations.

•Multiple failure modes are badly modeled as the response surface may not find all important regions (Waarts 2000).

The DARS method is a combination of the direct and indirect method (Waarts 2000). It can be looked at as an algorithm minimizing the number of direct limit state function evaluations. Each evaluation requires a call to the external program Calipous. The DARS procedure is performed in different steps, see Figure 1, illustrated for a mathematical example in the standard normal space (u-space) with two random variables:

Step 1. In a first step the value of each random variable is changed individually until the root (λ) of the limit state function is found. First a linear estimate is calculated, based on the outcome (LSFE) in the origin of the u-space (standard normal space) $(0,0,0,\ldots,0)$ and the point $(0,0,\ldots,\pm f,\ldots,0)$. When known, the factor f is set equal to the expected reliability index β. Experience shows that $f=3$ performs good as well. The limit state function is evaluated in this first approximation. Further approximations are based on a quadratic fit through the outcome (LSFE) of the different iteration points. Each time extra verification is done to assure convergence to the correct root in the preset direction. Mostly after 3 to 4 iterations (LSFE) the root is found (limit of safe domain), assumed there is a root in the specified direction, Figure 1, left part.

Step 2. An initial response surface (RS_1) is fit through the data using a least square algorithm, Figure 1, mid part.

Step 3. This step is an iterative procedure, Figure 1, right part. The response surface is adapted (Adaptive Response Surface method (ARS)) and the failure probability or reliability index are updated until the required accuracy is reached. Therefor, coarse directional sampling is performed on the response surface. For each sample, a first estimate of the distance $\lambda_{i,RS,i-1}$ to the origin in the u-space is made based on the response surface (RS_{i-1}). An arbitrary distance λ_{add} is used to make distinction between important and less important directions: $\lambda_{i,RS,i-1} <> \lambda_{\min} + \lambda_{\mathrm{add}}$, in which $\lambda_{\min}$ is the minimum distance found so far. When λ_{add} is set equal to 3, a sufficient accuracy is reached within a relatively small number of samples. The value $\lambda_{\mathrm{add}} = 3$ is an arbitrary value that is optimized for the case of probabilistic design, where target β_T values around 3.8 are aimed at. In case the obtained root ($\lambda_{i,RS}$) has a relatively high contribution - $\lambda_{i,RS,i-1} < \lambda_{\min} + \lambda_{\mathrm{add}}$ - to the estimated global failure probability (p_f), the root of the real response is used in stead of the response surface: $\lambda_{i,LSFE}$. This requires several extra limit state function evaluations using the limit analysis Calipous. The response surface is updated with these new data from the moment they are available ($RS_{i-1} \Rightarrow RS_i$). In case the contribution is less important - $\lambda_{i,RS,i-1} > \lambda_{\min} + \lambda_{\mathrm{add}}$ - the contribution based on the root of the response surface ($\lambda_{i,RS,i-1}$) is used, avoiding time consuming limit state function evaluations (LSFE).

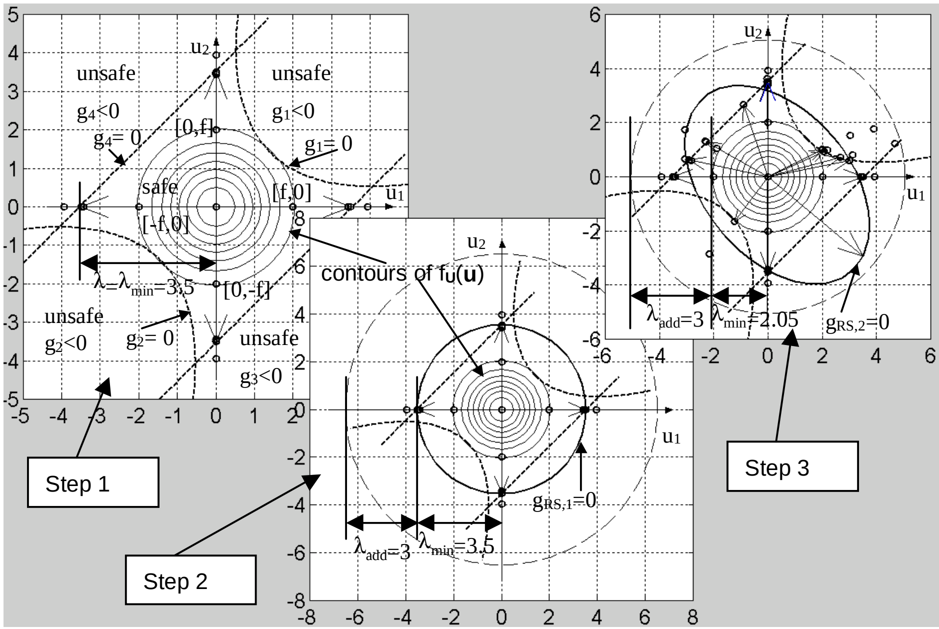

Figure 1. DARS - step 1-3

It is proved that the expected value $\left(E\left(\hat{p}_f\right)\right)$ of all contributions ($\hat{p}_i$) is an unbiased estimate of the global failure probability p_f (Melcher, 1999; Waarts, 2000):

$$p_f \approx E\left(\hat{p}_f\right) = \frac{1}{N}\sum_{i=1}^{N}\hat{p}_i, \quad where\colon\ \hat{p}_i = \chi_n^2\left(\lambda_{i,RS\ or\ LSFE}\right) \tag{2}$$

The major advantage of the method lies in the number of direct limit state function evaluations. These remain proportional to the number of random variables (n). Waarts estimated that the number of direct limit state function evaluations is limited to approximately $15n$ for an acceptable accuracy: $V(\beta)=0.05$ (Waarts, 2000). For low reliability values, $\sigma(\beta)=0.15$ is proposed as stop criterium. This linear increase gains interest for high numbers of n. In addition, there is no preference for a certain failure mode. All contributing failure modes are accounted for, resulting in a safety value that includes the system behaviour, thus on level III.

For purpose of this research, the DARS procedure was implemented in Matlab 5.3 (Matlab 2000). In combination with an automatic interface with the Calipous program (Smars 2000) this provides a powerful tool to calculate the reliability of masonry arches at level III.

SAFETY OF ARCHES

To evaluate the arch safety for a given set of parameters, the thrust line method is used (Heyman 1982). It is a limit analysis (LA) method using the equation of equilibrium and the resistance characteristics of the materials. In the case of arches, it is usually supposed that: blocs are infinitely resistant, joints resist infinitely to compression, joints do not resist to traction and joints resist infinitely to shear. These hypotheses are certainly restrictive: the material(s) used to construct the arch do not respect them strictly. It was nevertheless shown that - under normal circumstances - they are reasonable (Heyman 1982). The limits of this theory are discussed elsewhere (Smars 2000).

Typical failure modes associated with the limit situations are represented in Figure 2.

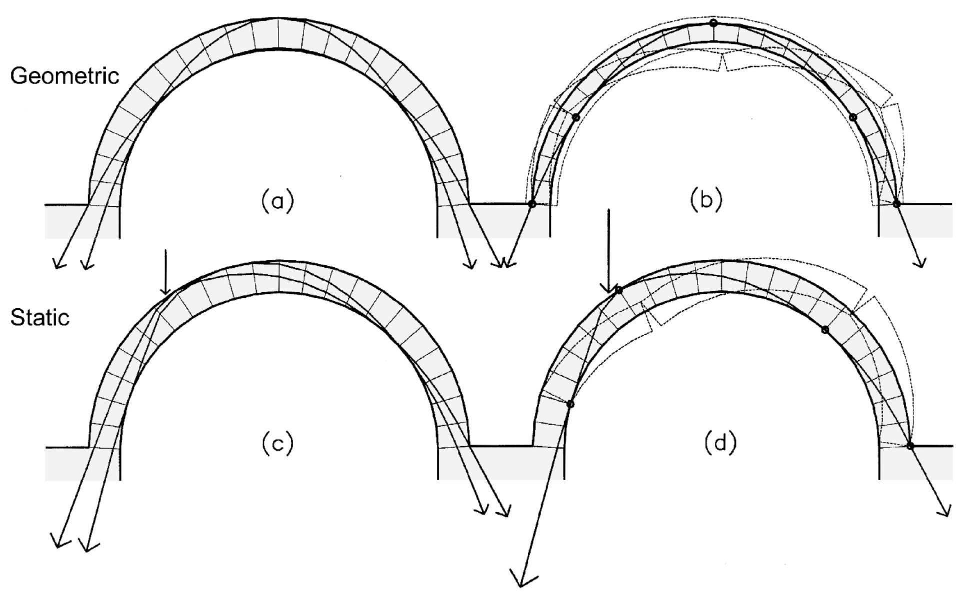

Figure 2. Arches - Failure modes and safety factors α_g and α_S

Given these hypotheses, it can be shown (Kooharian 1952) that an arch is stable if a thrust line, remaining entirely inside its shape, can be found. Analytical expressions relating parameters to stability are not available for generic situations. The code Calipous was developed to compute numerical estimates. In particular, it can determine safety factors. The geometrical factor of safety (α_g) is defined as the minimal multiplicative factor on the arch thickness allowing an internal thrust line to be found. The static factor of safety (α_s) is defined as the maximal multiplicative factor on external forces allowing an internal thrust line to be found.

As long as the safety factors exceed one, the structure is in the safe region, the arch is stable. If one of the safety factors is smaller than one, the structure is in the unsafe region, the arch is unstable. This results in the following limit state functions:

$$\begin{aligned} g_1(\mathbf{X}) &= \alpha_g(\mathbf{X}) - 1 \\ g_2(\mathbf{X}) &= \alpha_S(\mathbf{X}) - 1 \end{aligned} \tag{3}$$

SAFETY ASSESSMENT OF AN EXISTING MASONRY ARCH

As an example, the global failure probability of a masonry arch, Figure 3, will be determined. Besides, possible actions in case of insufficient safety will be discussed. A fictitious example was designed to illustrate the method.

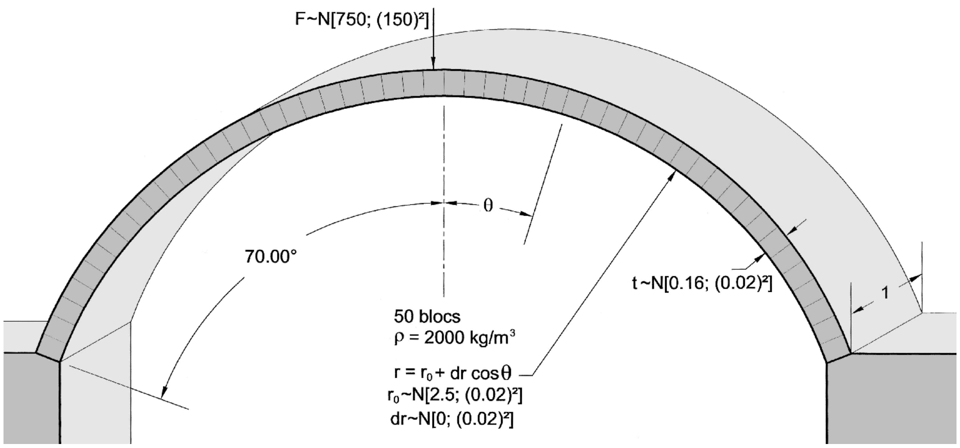

Figure 3. Fictitious existing masonry arch

The situation (geometry and forces) is chosen so as to lead to a low geometrical safety factor ($\alpha_g = 1.23$). The arch being not far from instability, the uncertainties on geometry start to play a role. The geometry indeed is not exactly known. A function is chosen to approach the possible variations. Its parameters are: (1) the mean diameter of the arch r_0, (2) its thickness t, (3) the 2^{nd} order deviation with respect to a perfect circular arch dr and (4) an eccentric vertical load F, applied in the most critical position. These parameters are a translation of the uncertainties that might occur during a survey of the existing arch, Table 3. In practice, one needs to choose them carefully. If the important ones are neglected, the evaluation of the failure probability will yield to non-significant results. Indeed, in some cases, other factors could be of importance (local geometrical faults for instance).

The force application point on the arch was chosen to minimize the geometrical and static safety factors on the perfect geometry. For this particular example, the standard deviations of the different random variables were chosen arbitrarily but in a real case these should be deduced from the measured geometry accounting for the survey technique(s) and for the force, using an available code.

Table 3. Random variables and their parameters

Random variables	Probability density function	Mean value μ	Standard deviation σ	Coefficient of variation *V* [%]
$x_1 = r_0\ [m]$	Normal	2.5	0.02	0.4
$x_2 = t\ [m]$	Normal	0.16	0.02	17
$x_3 = dr\ [m]$	Normal	0.0	0.02	/
$x_4 = F\ [N]$	Lognormal	750	150	20

The mean value of these parameters results in the following safety factors:
•geometrical safety factor: $\alpha_g = 1.23$,
•static factor of safety: $\alpha_s = 2.39$.

An interface between the DARS optimization algorithm and the Calipous program was devised to allow automatic processing. The standard deviation ($\sigma(ß)$) or the coefficient of variation ($V(ß)$) on the reliability index can be used as a criterion to stop the sampling process: $\sigma(ß)=0.15$ or $V(ß)=0.05$. As the processing time for an analysis using Calipous is tolerably small (15 up to 20 seconds per analysis), the focus was not put on optimal time efficiency and the number of samples *N* was just set to 500 for illustrative purposes. However, it is marked where the preset accuracy is reached. The results are summarized in Table 4. The computer time (CPU: Pentium 667 MHz, RAM: 128 MB) and number of limit state function evaluations (LSFE) are listed. The number of samples *N* and LSFE needed to reach the preset accuracy are mentioned too.

Table 4. Results for the initial analysis

Results	β	$\sigma(\beta)$	p_f	*N*	*CPU*	*LSFE*
DARS analysis using limit analysis (Calipous)	1.26 1.21	**0.15** 0.14	$1\ 10^{-1}$ $1.1\ 10^{-1}$	371 500	50 min 67 min	149 202
Values based on the estima- ted Response Surface g_{RS}	0.93	0.002	$1.8\ 10^{-1}$	$5\ 10^5$	8.0 sec	/
Direction cosines α, based on the Response Surface g_{RS}	α_1 (r) -0.03	α_2 (t) 0.94	α_3 (dr) -0.11	α_4 (F) -0.32		

To reach the preset accuracy $\sigma(\beta)=0.15$, 371 samples and 149 LSFE are required. This position is marked on Figure 4. Remark that these 371 samples did only require 149 direct limit state function evaluations or calls to the Calipous program, although this is higher than $15n = 60$ as preset by Waarts (Waarts 2000). For the other samples, the estimated response surface was accurate enough to replace time-consuming direct limit state function evaluations. After 500 samples, the standard deviation on the reliability index $\sigma(\beta)=0.14$. Coarse directional sampling would require 2 hours and 10 minutes for 500 samples.

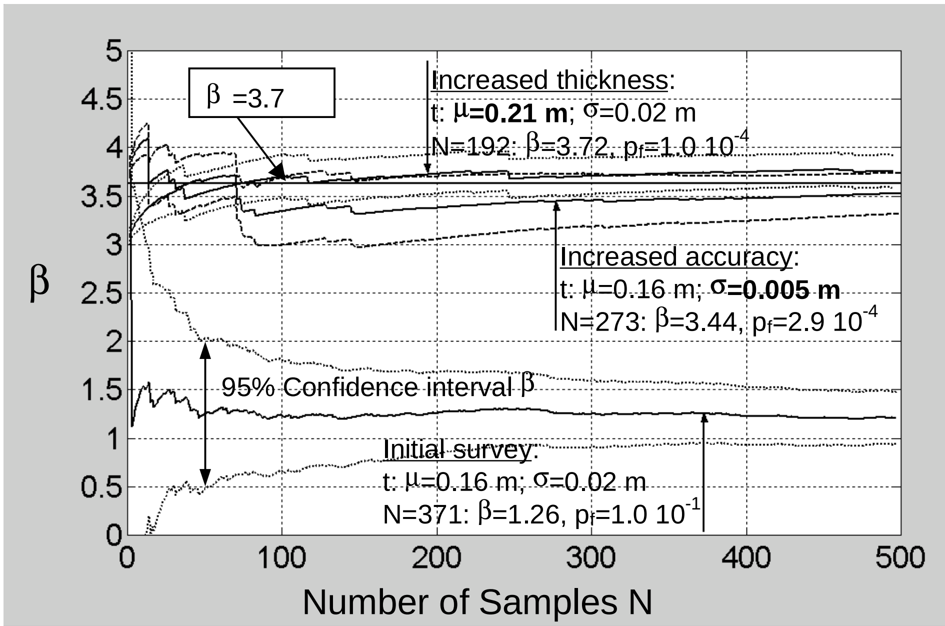

Figure 4. DARS outcome - Reliability β versus number of Samples N

For the response surface, a polynomial of second order was used. As the form of the limit state function is not known beforehand, its functional form has to be estimated. For most structural problems a polynomial of first or second order is appropriate. In other cases a more complex relation has to be searched for (Montgomery 1997). To illustrate that the response of the arch can be represented accurately for a wide range of the stochastic variables, the probability of failure is calculated by means of a Crude Monte Carlo analysis based on the final estimate of the response surface:

$$\begin{aligned} g_{RS} = & -0.1467 - 0.3637x_1 + 9.5282x_2 - 0.9977x_3 - 0.004x_4 \\ & + 0.0086x_1^2 - 1.8373x_2^2 - 0.6831x_3^2 + 0.0000x_4^2 \end{aligned} \quad (4)$$

This result is added in the lower part of Table 4. Because of the arithmetic simplicity of the second order polynomial, only 8 seconds of CPU are required to perform the 500.000 Monte Carlo simulations on the response surface (ß =0.93). The value based on the response surface underestimates the real reliability index. This emphasizes the importance of using the real outcomes of the limit analysis (Calipous) in the important regions.

The 95% confidence interval for the reliability index ß and global failure probability p_f are: 95% *CI(ß) = [0.98;1.44]* or 95% *CI(p_f)= [0.07;0.13]*. The evolution of the reliability index as a function of the number of simulations is presented in Figure 4. Upper- and lower boundaries of the 95% confidence interval are shown in dashed line.

The previous result indicates that the structure does not meet the target reliability index according to the European Standard Eurocode 1 (EC1, 1994): $ß_T = 3.7$. To meet the safety requirements, two options can be taken: (1) perform a more accurate survey of the arch or (2) consolidate the structure. In both cases it is interesting to determine the most critical parameters with respect to the failure probability in order to maximize the efficiency of the action(s) taken. Naturally, if the safety was considered sufficient, then nothing should be done.

The direction cosines of the final estimate of the response surface at the failure (or design) point (Table 4: α_1, ..., α_4) are measures of the relative influences of the parameters. Here, the thickness of the arch is the most important parameter ($\alpha_2 = 0.94$). The gain of a more accurate survey of the thickness can be evaluated quickly using the response surface. Thus, if the standard deviation could be limited to $\sigma(t) = 0.005\ m$ (instead of the original value of 0.02 m), the response surface directly gives a first estimate of the probability of failure: $ß = 3.0$ *or* $p_f = 1.3\ 10^{-3}$. Observing that the Response Surface tends to underestimate the reliability index, a new analysis is performed to update this first estimate. These data are added in Table 5 and on Figure 4. Indeed, the real reliability index $ß$ = 3.53 is higher than the value estimated based on the response surface. It is interesting to see that the reliability index $ß$ = 3.53 almost equals the target reliability index $ß_T$ = 3.7-3.8. If it is decided that this is an acceptable safety level, further consolidation would no longer be required, leaving the structure maximally unaffected, in its authentic state. At this point, the importance of the choice of parameters again must be stressed. The parameter “time” could be very important: the abutments may settle inducing changes in the global geometry. If this is the case, the initial model might no longer cover the real structural behavior and the outcome would no longer be valid. If so, monitoring of these settlements can be used to ensure the validity of the chosen set of parameters.

Table 5. DARS analysis increased accuracy or thickness

Results	β	$V(\beta)$	p_f	N	CPU	$LSFE$
DARS analysis increased accuracy t: (μ = 0.16, **σ = 0.005 m**)	3.44 3.53	**0.05** 0.03	$2.9\ 10^{-4}$ $2.1\ 10^{-4}$	273 500	22 min 41 min	67 122
DARS analysis increased thickness t: (**μ = 0.21**, σ = 0.02 m)	3.72 3.75	**0.05** 0.02	$1.0\ 10^{-4}$ $0.9\ 10^{-4}$	192 500	16 min 52 min	48 126

If it is decided not to perform more accurate surveys for reasons of accessibility, time or expenses, then a consolidation would become necessary. One could for instance look for the required mean thickness of the arch to achieve the standard reliability of $\beta_T=3.7$ preset in EC1. This is most probably not a realistic reinforcement project, but it can exemplify the use of the method to calibrate interventions. Again the response surface leads to a first estimate: $\mu(t)=0.21\ m$. The reliability index increases to: β=3.75 or: $p_f=0.9\ 10^{-4}$, see Figure 4 and Table 5. The required number of samples and limit analyses is added in Table 5.

CONCLUSIONS

This paper deals with level III methods able to calculate the global failure probability of a structural system. The method is illustrated for the safety assessment of an existing masonry arch. Based on a first survey of the arch, an estimate of its safety can be calculated, accounting for the uncertainty of the measurements. Based on this analysis the most sensitive parameter(s) with respect to failure can be determined. The effect of possible interventions can be estimated a priori. For the present example of the masonry arch, this means that the initial low safety is mainly due to an inaccurate survey. If a more accurate survey of the thickness (the most critical parameter) can be performed, the extra knowledge would lead to a significant decrease of the estimated failure probability. In doing so, the monument remains intact, keeping its authenticity. If an intervention is necessary, optimal parameters can be estimated just to reach a target reliability index as preset in standard design codes.

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