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A CONSTITUTIVE MODEL FOR MASONRY WITH INTERNAL VARIABLES

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ABSTRACT

Three different constitutive models for the overall behaviour of masonry are presented. The no-tension model, the elastoplastic no-tension one and the damage behaviour coupled with elastoplasticity are cast in the unitary framework of the generalized standard material theory.

INTRODUCTION

At present a great deal of attention is being paid to the study of masonry constitutive behaviour. A considerable number of models has been proposed in the last twenty years for the in-plane behaviour of masonry, ranging from phenomenological to theoretical and numerical ones [Dialer, 1991; Ganz, 1989; Nappi et al., 1990] In the first class failure criteria, based on the mechanical interpretation of phenomena observed in laboratory tests, are included.
Constitutive models based on the mechanical modelling of masonry's peculiar characteristics, such as the elastic no-tension one, have been proposed and can be considered as theoretical approaches.
Numerical models, based mainly on FEM method, have been extensively developed in recent years, with the availability of high speed computers. Two main approaches may be considered: an arrangement reproducing the actual texture of masonry, considering separately units and mortar joints (micromodelling) [Arya et al., 1978], and a second one attempting to define an equivalent continuum, governed by constitutive laws capable of representing the global behaviour of masonry (macromodelling) [Page et al., 1985]. The former can be used to model a very small number of blocks and mortar joints, while the latter seems more suitable to analyze large masonry elements. Numerical models

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involving both micro or macromodelling and damage models have been recently proposed [Pietruszczak et al., 1992; Maksoud et al., 1992].
In this paper a general approach, based on internal variables approach, according to the generalized standard material theory [Halphen et al., 1975], is presented to model the behaviour of masonry.
In the first part the constitutive problem relative to a material which does not support tension is presented [Del Piero, 1989; Romano et al., 1979]. The no-tension behaviour is a drastic assumption, but can be improved by considering that the masonry's behaviour involves phenomena influenced by dissipation. The second part of the paper describe a constitutive model involving an elastoplastic and no-tension behaviour. The third section deals with an elastoplastic model with damage, suitable to represent the mortar behaviour. An homogenization procedure for the assemblage of mortar and bricks, together with a numerical procedure, will be the argument of a forthcoming paper.

NO-TENSION MODEL

The constitutive problem for a linear elastic material which does not support tension can be defined by the following relations:

$$\begin{cases} \sigma = C(\varepsilon - \delta) \\ < \sigma, \delta > = 0 \ , \quad \sigma \in Q \ , \quad \delta \in \Delta \end{cases}$$

where:

ε	total infinitesimal strain
δ	fracture strain
σ	infinitesimal stress
Δ	convex cone of admissible fracture strains
$Q = \Delta^-$	convex cone of admissible stresses
C	elasticity operator

Denoting by D the strain space and by S the dual space of stress, the cone of admissible stesses is:

$$Q = \Delta^- = \{ \sigma \in S : < \sigma, \delta > \leq 0, \forall \delta \in \Delta \}$$

where Δ^- is the negative convex polar cone of Δ. A two dimensional example is given in fig 1.

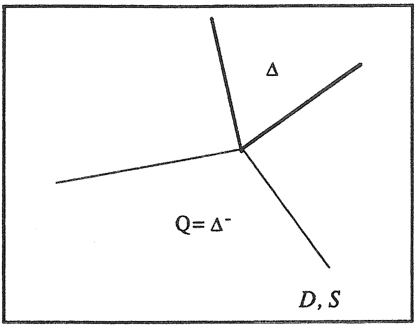

Fig. 1 Negative polar cone of a convex Δ

For the Cauchy model, the convex cone of admissible stresses Q can be defined in terms of the eigenvalues σ_i of the stress tensor in the form:

$$Q = \{ \sigma: \sigma_{imax} \leq 0 \}$$

Remark 1

In the constitutive problem it is assumed that the infinitesimal strain tensor ε can be decomposed into the sum of an elastic part e and an *anelastic* part δ, so that it turns out to be $\varepsilon = e + \delta$

Remark 2

The orthogonality between the pair (σ, δ) and the conditions $\sigma \in Q$, $\delta \in \Delta$ can be written alternatively as follows:

$$\sigma \in N_\Delta(\delta) = \partial \amalg_\Delta(\delta)$$

where $N_\Delta(\delta)$ is the normal cone to the admissible fracture convex cone Δ at the point δ and $\partial \amalg_\Delta(\delta)$ is the subdifferential of the indicator function $\amalg$ of Δ at the point δ (see App. A). Thus the no-tension constitutive model is defined by:

$$\begin{cases} \varepsilon = e + \delta \\ \sigma = Ce \\ \sigma \in \partial \amalg_\Delta(\delta) \end{cases}$$

This model involves one multi-valued relation, so that the related variational formulation can be derived by appealing to the potential theory for monotone multivalued operators [Romano et al., 1993]. The no-tension model can be written in an operational form by defining the multi-valued operator T from the product space $S \times D \times D$ to the product space $S \times D \times D$:

$$(0,0,0) \in \mathbf{T}(\sigma,\delta,e) + (\varepsilon,0,0)$$

or, explicitly

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} & -I_D & -I_D \\ -I_S & \partial C_\Delta & \\ -I_S & & C \end{bmatrix} \begin{bmatrix} \sigma \\ \delta \\ e \end{bmatrix} + \begin{bmatrix} \varepsilon \\ 0 \\ 0 \end{bmatrix}$$

where I_D and I_S are respectively the identity in the spaces D and S.
It can be proved [Romano et al., 1993] that the operator **T** is conservative, thus its potential is given by the sum of the potentials relative to the single operators in which T can be decomposed. Hence it turns out to be:

$$\Omega_f(\sigma,\delta,e) = \varphi(e) + \amalg_\Delta(\delta) - < \sigma, e + \delta > + <\sigma, \varepsilon >$$

where $\varphi(e) = 1/2 <Ce , e>$ is the elastic energy and $\amalg_\Delta(\delta)$ is the potential of $\partial\amalg_\Delta(\delta)$.

As a consequence, a vector (σ,δ,e) is a solution of the no-tension constitutive model for a fixed ε if, and only if, it minimizes the potential Ω_f [Romano et al, 1993; Marotti de Sciarra, 1994].

ELASTOPLASTIC AND NO-TENSION MODEL

A schematization of masonry, considered as an homogeneous body, can be provided by assuming that the model has a null tensile strength and an elastoplastic behaviour with hardening in compression. The hardening behaviour is introduced in the constitutive no-tension model following the generalized standard material theory (Halphen and Nguyen, 1975), based on the definition of internal state variables. The generalized stress and strain vectors are defined in the product spaces:

$$\mathbb{D} = D \times X \qquad\qquad \mathbb{S} = S \times X'$$

where X and X' are the spaces of internal kinematic and dual static variables, respectively. The generalized vectors are defined by:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon \\ \alpha \end{bmatrix} \qquad \mathbf{e} = \begin{bmatrix} e \\ 0 \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma \\ \chi \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} d \\ -\alpha \end{bmatrix}$$

where:

α	internal kinematic variable	χ	internal static variable
$\boldsymbol{\varepsilon}$	generalized total strain	$\boldsymbol{\sigma}$	generalized stress
$\mathbf{e}$	generalized elastic strain	$\mathbf{d}$	generalized *anelastic* strain

The admissible convex cone of generalized internal forces must be defined in the generalized space $\mathcal{S}$:

$$\mathbb{Q} = \{ (\sigma; \chi) \in \mathcal{S} : \langle \sigma, d \rangle - \langle \chi, \alpha \rangle \leq 0, \forall \delta \in \Delta ; \forall \chi \in X' \}$$

Denoting by $\mathbb{C} \subseteq \mathcal{D}$ the generalized convex elastic domain, the elastic-non fracturing domain $\mathbb{V}$ can be defined as the intersection between $\mathbb{C}$ and $\mathbb{Q}$: $\mathbb{V} = \mathbb{C} \cap \mathcal{D}$. A schematic picture is given in fig. 2. Note that the set $\mathbb{V}$ is convex, being the intersection of two convex sets. The normality rule to the set $\mathbb{V}$ can be stated considering that:

- a generalized *plastic* flow ($\dot{p}$, $-\dot{\alpha}$)$\in \mathbb{V}$ can be associated with a generalized stress which belongs to the boundary of the elastic domain $\mathbb{C}$;

- a generalized *fracture* flow ($\dot{\delta}$, $-\dot{\alpha}$)$\in \mathbb{D}$ can be associated with a generalized stress which belongs to the boundary of the admissible cone $\mathbb{Q}$.

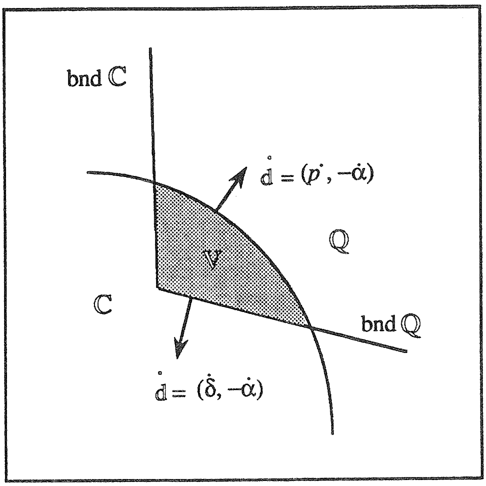

Fig. 2 Elastic-non-fracturing domain V

Thus, it can be stated that:

$$\begin{cases} \sigma \in \mathbb{Q} \cap \text{bnd } \mathbb{C} \Rightarrow \dot{d} = (\dot{p}, -\dot{\alpha}) \\ \sigma \in \mathbb{C} \cap \text{bnd } \mathbb{Q} \Rightarrow \dot{d} = (\dot{\delta}, -\dot{\alpha}) \end{cases}$$

The normality assumption can be written:

$$\dot{d} \in N_{\mathbb{V}}(\sigma) = N_{\mathbb{C} \cap \mathbb{Q}}(\sigma, \chi)$$

Being $\mathbb{V}$ convex, the normal cone $N_{\mathbb{V}}$ can be equivalently written in terms of the indicator of $\mathbb{V}$ in the form:

$$\dot{d} \in \partial \amalg_{\mathbb{V}} (\sigma)$$

The generalized total strain can be decomposed into an elastic and an *anelastic* part as follows:

$$\varepsilon = e + d$$

The generalized elastic constitutive relation can be stated by considering the convex free energy ϕ, defined in the generalized strain space $\mathcal{D}$:

$$\sigma = d\phi(e)$$

The constitutive problem for an elastoplastic material which does not support tension is defined by:

$$\begin{cases} \varepsilon = e + d \\ \dot{d} \in \partial \amalg_{\mathbb{V}} (\sigma) \\ \sigma = d\phi(e) \end{cases}$$

The generalized flow rule can be equivalently written in terms of the conjugate potential $\amalg^*_{\mathbb{V}}$ of $C_{\mathbb{V}}$ in the form:

$$\sigma \in \partial \amalg^*_{\mathbb{V}} (\dot{d})$$

Time integration of the above flow rule is performed according to the fully implicit backward Euler integration scheme, because of its simplicity and unconditional stability. Hence the generalized anelastic strain rate becomes:

$$\dot{d} \cong \frac{d - d_0}{\Delta t}$$

where Δt is the time interval and d_0 is the initial value of d. Since Δt is a positive scalar, the normality rule becomes:

$$d - d_0 \in \partial N_{\mathbb{V}} (\sigma) \Leftrightarrow \sigma \in \partial \amalg^*_{\mathbb{V}} (d - d_0)$$

The constitutive problem for an elastoplastic material which does not support tension can be written in operational form as follows:

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} & -I_D & -I_D \\ -I_S & \partial \Pi_V^* & \\ -I_S & & d\phi \end{bmatrix} \begin{bmatrix} \sigma \\ d - d_0 \\ e \end{bmatrix} + \begin{bmatrix} \varepsilon \\ 0 \\ 0 \end{bmatrix}
$$

Hence the potential turns out to be:

$$
\Omega_p(\sigma, d, e) = \varphi(e) + \Pi_V^*(d - d_0) - < \sigma, e + d > + < \sigma, \varepsilon >
$$

where $\varphi(e)$ is the elastic energy and Π_V^* the potential of $\partial \Pi_V^*$.

As a consequence, a generalized vector (σ, d, e) is a solution of the elastoplastic no-tension constitutive model for a fixed ε if and only if it minimizes the potential Ω_p [Romano et al, 1993; Marotti de Sciarra, 1994].

ELASTOPLASTIC MODEL WITH DAMAGE

A different approach can be given to the study of masonry behaviour. The mortar is assumed to have an elastoplastic behaviour with hardening and damage. The brick can be considered elastic and the masonry can be modelled by an homogenization of the two phases [Swan et al., 1993]. The constitutive problem for an elastoplastic material with damage can be stated considering a generalized standard material with an internal variable which governs the damage behaviour. Hence, let us define the dual spaces:

$$
\mathcal{D} = D \times X \times Y \qquad \qquad \mathcal{S} = S \times X' \times Y'
$$

where X and X' are the spaces of internal kinematic and static variables which take into account the hardening phenomena, Y and Y' are the space of internal kinematic and static variables which take into account damage, respectively. The generalized vectors are defined by:

$$
\varepsilon = \begin{bmatrix} \varepsilon \\ \alpha \end{bmatrix} \qquad e = \begin{bmatrix} e \\ 0 \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma \\ \chi \end{bmatrix} \qquad p = \begin{bmatrix} p \\ -\alpha \end{bmatrix}
$$

and, in addition, two damage variables are introduced: $\eta \in Y$ is the internal damage kinematic variable and $\xi \in Y'$ is the dual static internal variable.

Denoting by $\mathbb{C} \subseteq \mathbb{D}$ the generalized elastic domain and by $H \subset Y'$ the convex set of static internal variables ξ, the following evolutive relations are assumed:

$$
\dot{p} \in N_c(\sigma) \Leftrightarrow (\dot{p}, -\dot{\alpha}) \in N_c(\sigma, \chi) = \partial \Pi_c(\sigma, \chi)
$$

$$\dot{\eta} \in N_H(\xi) = \partial \sqcup\!\!\sqcup_H(\xi)$$

Note that if ξ belongs to the interior of H the rate of the kinematic damage variable $\dot{\eta}$ is vanishing, hence there is no evolution of the damage. The free energy is a differentiable convex functional defined in the product space $D \times Y$ and, in this case, is given by:

$$\Phi(e,\eta) = (1 - \eta)\, \varphi(e) + \psi(\alpha)$$

The convex functional φ denotes the elastic energy, while the convex functional ψ accounts for the hardening phaenomena. The constitutive problem for an elastoplastic material with damage is defined by:

$$\begin{cases} \varepsilon = e + p \\ \dot{\eta} \in N_H(\xi) \\ \dot{p} \in N_c(\sigma) \\ \sigma = d_e\Phi(e,\eta) \\ \xi = d_\eta\Phi(e,\eta) \end{cases}$$

Remark
Note that the static internal variable ξ coincides with the opposite of the elastic energy $\varphi(e)$ [Simo et al., 1987]

$$\xi = -\varphi(e)$$

The finite-step counterpart of the constitutive model above is obtained according to the fully implicit backward rule scheme:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} & -I_D & -I_D & & \\ -I_S & \partial \sqcup\!\!\sqcup^*_\Delta & & & \\ -I_S & & & & \\ & & d\Phi & & -I_Y \\ & & & -I_Y & \partial \sqcup\!\!\sqcup_H \end{bmatrix} \begin{bmatrix} \sigma \\ p - p_0 \\ e \\ \eta \\ \xi \end{bmatrix} + \begin{bmatrix} \varepsilon - p_0 \\ 0 \\ 0 \\ 0 \\ \eta_0 \end{bmatrix}$$

So that the potential is given by:

$$\Omega_p(\sigma, p, e, \eta, \xi) = \varphi(e, \eta) + \amalg_V^*(p - p_0) + \amalg_H(\xi) - < \sigma, e+p > + < \sigma, \varepsilon > - \xi(\eta - \eta_0)$$

CONCLUSIONS

The elastoplastic no-tension model would be a good approach to represent the global behaviour of masonry, but a suitable description of the elastic domain and its evolution is not available. Hence a different approach can be performed: mortar and bricks can be modelled separately and the global behaviour of masonry can be obtained as a result of an homogenization procedure.

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Appendix A. Elements of convex analysis

The *subdifferential* $\partial f: X \rightarrow X'$ is a multivalued map which associates with every vector x the convex set of vectors x' defined by:

$$\partial f(x) = \{ x' \in X' : f(z) - f(x) \geq < x', z-x > \forall z \in X \}$$

A *cone* Δ is the convex set of vectors δ defined by:

$$\delta \in \Delta \Leftrightarrow \lambda\delta \in \Delta$$

where λ is a nonnegative scalar.

The *normal cone* to a convex set $C \subseteq X$ at x is:

$$N_C(x) = \begin{cases} \{ x' \in X' : < x', x_o - x > \leq 0, \forall x_o \in X \} \text{ if } x \in C \\ \varnothing \text{ if } x \notin C \end{cases}$$

The *indicator function* of a convex $\amalg$ is:

$$\amalg_C(x) = \begin{cases} 0 \quad \text{if } x \in C \\ +\infty \quad \text{if } x \notin C \end{cases}$$

The *subdifferential of the indicator* function of a convex set is defined by:

$$\partial\amalg_C(x) = \begin{cases} \{ x' \in Q : < x', x_o - x > \leq 0, \forall x_o \in X \} \text{ if } x \in C \\ \varnothing \text{ if } x \notin C \end{cases}$$

which coincides with the normal cone N_C at the point x:

$$\partial\amalg_C(x) = N_C(x)$$