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EVALUATION OF COLLAPSE LOAD FOR MASONRY WALLS

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ABSTRACT

The authors take again a very simple formulation for determining the load collapse multiplier for masonry structures, by a linear formulation founded on classical limit analysis theorems. To draw the model as close as possible to the real collapse mechanism, it is now supposed that failure lines can also form along the diagonals of macroblocks, so introducing even triangular elements into the discretization process. Moreover, to underline the influence that the modality of the texture of bricks or stones has on the behaviour of a panel, the friction coefficient $tg\varphi_o$ used for the horizontal interfaces is suitably increased along the diagonal and vertical interfaces of the macroblocks.

KEYWORDS: limit analysis, macroblocks, triangular elements, friction coefficients

INTRODUCTION

Our research examines the safety of masonry panels in the presence of seismic forces through the theorems of limit analysis. Walls are shaped as systems of rectangular or triangular rigid macroblocks and are studied in the case of middle plane forces, with the assumption that on interfaces between the blocks there is inability to carry tension, unlimited compression resistance and sliding with dilatancy. The analysis of the mechanical behaviour of a masonry structure with the above-mentioned assumptions has been the subject of a rich literature. Recently the problem has also been dealt with through nonlinear programming, by supposition of not associated frictional sliding [1, 2].

We have presented an alternative to the above-mentioned procedures - often too expensive and in any case unable to guarantee the uniqueness of the solution [3]. The simple method proposed is founded on the static theorem of limit analysis, and makes use of Excel's *solver* to fix the load collapse multiplier for masonry structures. The first results obtained were in accordance with the ones achieved by other researchers.

Here, the same formulation is re-proposed with the aim of improving the results just achieved for masonry walls. Therefore, to draw the models as close as possible to the real collapse mechanism, it is now supposed that failure lines can also form along the diagonals of macroblocks, so introducing even triangular elements into the discretization process. Moreover, to underline the influence that the modality of the texture of the bricks or stones has on the behaviour of a panel, the friction coefficient $tg\varphi_o$ used for the horizontal interfaces is suitably increased along the diagonal and vertical interfaces of the macroblocks.

THE EQUILIBRIUM CONDITION FOR THE SINGLE BLOCK

We will refer to a generic plane masonry panel, discretized by rectangular or triangular large blocks (Figure 1).

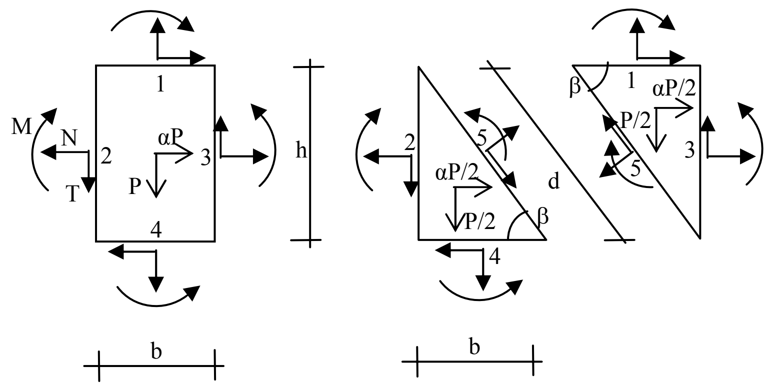


Figure 1

The contact forces N, T, M on the interfaces are supposed to be applied at the centroid of each interface.

Therefore every block is generally subject to resultants N, T, M on the interfaces and to the dead and horizontal live loads applied at the centroid of the same block (P is the self weight of the block).

The equilibrium equations of a generic rectangular or triangular block can be briefly expressed with the form:

$$\mathbf{A}^e \mathbf{X}^e + \mathbf{F}_v^e + \alpha\, \mathbf{F}_o^e = \mathbf{0} \qquad \text{Equation 1}$$

where "e" is the element's index, $\mathbf{A}^e$ is a matrix (3x12) or (3x9) for the rectangular and triangular elements respectively, depending on the dimensions of the block, $\mathbf{X}^e$ is the vector of the unknown stress resultants on the interfaces, $\mathbf{F}_v^e$ is the vector of the dead loads and $\alpha\mathbf{F}_o^e$ the vector of the horizontal live loads, with α being an unknown multiplier.

YIELD DOMAIN FOR THE GENERIC INTERFACE

The stress resultants on the interfaces have to respect the yield domains of the material for sliding and rocking (Figures 2a, 2b).

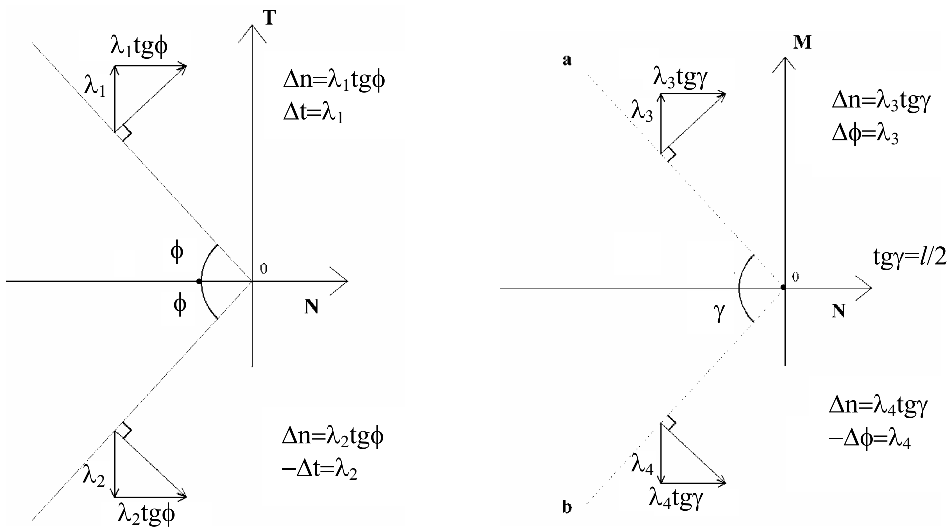


Figure 2 – a) Limit surface for sliding; b) Limit surface for rocking

Having supposed an unlimited compressive strength, four conditions have to be imposed on every generic interface; in the matrix form they are:

$$\begin{bmatrix} \mathrm{tg}\varphi & 1 & 0 \\ \mathrm{tg}\varphi & -1 & 0 \\ l/2 & 0 & 1 \\ l/2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{N} \\ \mathrm{T} \\ \mathrm{M} \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Equation 2}$$

$$\mathbf{D_i}\ \mathbf{X_i} \leq 0 \qquad \text{Equation 3}$$

where φ is the generic angle of friction (φ_o, φ_v, φ_d refer to horizontal, vertical or diagonal interfaces respectively) and l is a generic length of the interface (b, h, d in Figure 1).
Particularly, in the case of diagonal interface, the analysis of a block formed by two triangular elements, subject to dead weight and horizontal live loads, shows that the more suitable values of the friction coefficient $\mathrm{tg}\varphi_d$ have to be not lower than $\mathrm{tg}\varphi_d^* = (1+\sin^2\beta)/(\sin\beta\ \cos\beta)$, corresponding to the failure due to sliding and rocking on the diagonal (Figure 3).

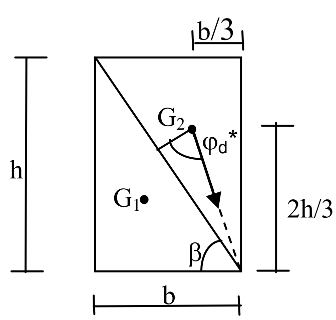


Figure 3

GOVERNING CONDITIONS

If *n* and *m* are the numbers of blocks and interfaces, the equilibrium conditions are:

$$\mathbf{A}\,\mathbf{X} + \mathbf{F_v} + \alpha\,\mathbf{F_o} = \mathbf{0} \qquad \text{Equation 4}$$

and the yield domain's conditions are:

$$\mathbf{Y} = \mathbf{D}\,\mathbf{X} \leq 0 \qquad \text{Equation 5}$$

where $\mathbf{A}$ is a (3n x 3m) matrix, $\mathbf{X}$ is a 3m-vector, $\mathbf{F_v}$ and $\mathbf{F_o}$ are 3n-vectors, α is the unknown collapse multiplier, $\mathbf{D}$ is a (4m x 3m) matrix.

In operating terms it is better to split $\mathbf{X}$ into two sub-vectors $\mathbf{X_1}$ and $\mathbf{X_2}$, where the second 3 (m-n) sub-vector collects the "hyperstatic" unknowns. The introduction of diagonal lines in the rectangular blocks does not increase the number of these unknowns.

Consequently, the problem can be formulated in the following manner:
maximize α
subject to

$$\mathbf{A_1}\,\mathbf{X_1} + \mathbf{A_2}\,\mathbf{X_2} + \mathbf{F_v} + \alpha\,\mathbf{F_o} = \mathbf{0} \qquad \text{Equation 6}$$

$$\mathbf{Y} = \mathbf{D_1}\,\mathbf{X_1} + \mathbf{D_2}\,\mathbf{X_2} \leq 0 \qquad \text{Equation 7}$$

$$\alpha \geq 0 \qquad \text{Equation 8}$$

with $\mathbf{A_1}$ being a (3n x 3n) invertible matrix; or better, the problem can be reformulated in the unknowns $\mathbf{X_2}$ and α only, as:

maximize α
subject to

$$\mathbf{Y} = \mathbf{D'}\mathbf{X_2} - \mathbf{D_o}\,\alpha - \mathbf{D_v} \leq \mathbf{0} \qquad \text{Equation 9}$$

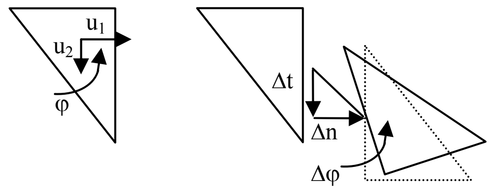
$$\alpha \geq 0 \qquad \text{Equation 10}$$

being: $\mathbf{D'} = \mathbf{D_2} - \mathbf{D_1}\mathbf{A_1}^{-1}\mathbf{A_2}$, $\mathbf{D_o} = \mathbf{D_1}\mathbf{A_1}^{-1}\mathbf{F_o}$, $\mathbf{D_v} = \mathbf{D_1}\mathbf{A_1}^{-1}\mathbf{F_v}$.

THE EVALUATION OF THE COLLAPSE MECHANISM

When the unknowns α and $\mathbf{X_2}$ have been defined, we can pursue the kinematic problem. The unknowns of this problem are:
- the vector $\mathbf{u}$ which collects the degrees of freedom (i.e., three in the centroid of every block, for the structure discretized (Figure 4a));
- the vector Δ which collects the displacements between the interfaces (i.e., three for every interface (Fig.4b), formed by sub-vectors $\mathbf{\Delta_1}$ and $\mathbf{\Delta_2}$ respectively corresponding to sub-vectors $\mathbf{X_1}$ and $\mathbf{X_2}$);

- the vector $\boldsymbol{\lambda}$ which collects the generalized strain rates associated to the yield conditions (i.e., four for every interface (Figures 2a, 2b)).



(a) **(b)**

Figure 4 - a) The degrees of freedom for a block; b) The displacements between the interfaces

These unknowns are bounded by kinematic conditions:

$$\mathbf{A}^{\mathbf{T}}\,\mathbf{u} = \boldsymbol{\Delta} \qquad \text{Equation 11}$$

and by the flow rule

$$\boldsymbol{\Delta} = \mathbf{D}^{\mathbf{T}}\,\boldsymbol{\lambda} \qquad \text{Equation 12}$$

These, opportunely split, give:

$$\mathbf{A_1}^{\mathbf{T}}\,\mathbf{u} = \mathbf{D_1}^{\mathbf{T}}\,\boldsymbol{\lambda} \qquad \text{Equation 13}$$

$$\mathbf{A_2}^{\mathbf{T}}\,\mathbf{u} = \mathbf{D_2}^{\mathbf{T}}\,\boldsymbol{\lambda} \qquad \text{Equation 14}$$

If we invert Equation 13 and we put the vector

$$\mathbf{u} = (\mathbf{A_1}^{\mathbf{T}})^{-1}\,\mathbf{D_1}^{\mathbf{T}}\,\boldsymbol{\lambda} \qquad \text{Equation 15}$$

in Equation 14 we have the kinematic conditions in the unknowns $\boldsymbol{\lambda}$ alone:

$$(\mathbf{A_2}^{\mathbf{T}}\,(\mathbf{A_1}^{\mathbf{T}})^{-1}\mathbf{D_1}^{\mathbf{T}} - \mathbf{D_2}^{\mathbf{T}})\,\boldsymbol{\lambda} = \mathbf{0} \qquad \text{Equation 16}$$

or

$$\mathbf{M}\,\boldsymbol{\lambda} = \mathbf{0} \qquad \text{Equation 17}$$

with $\mathbf{M} = \mathbf{A_2}^{\mathbf{T}}\,(\mathbf{A_1}^{\mathbf{T}})^{-1}\mathbf{D_1}^{\mathbf{T}} - \mathbf{D_2}^{\mathbf{T}}$.

Equation 17 can be simplified opportunely if we consider only the unknowns $\lambda_i \neq 0$ corresponding to the yield condition $Y_i = 0$ in Equation 9; so we obtain the solution as function of an arbitrary parameter, and the collapse mechanism by vector **u** in Equation 15.
We have also pursued this solution and its collapse configuration making use of *Excel*.

APPLICATIONS

To compare our results with those obtained - using linear and nonlinear programming - by other authors, we have analyzed three walls without (Figure 5a) and with openings (Figures 5b and 5c) already studied in [1, 2], having constant thickness, discretized in elements of size 4 x 1.75, and with friction coefficient 0.65. The computational results are reported in Tables 1, 2 and 3.

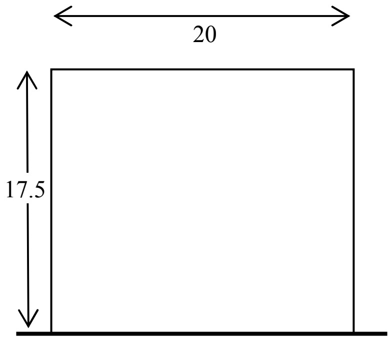


Figure 5a

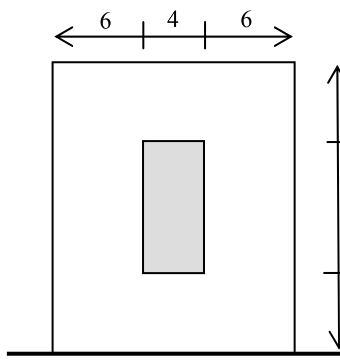


Figure 5b

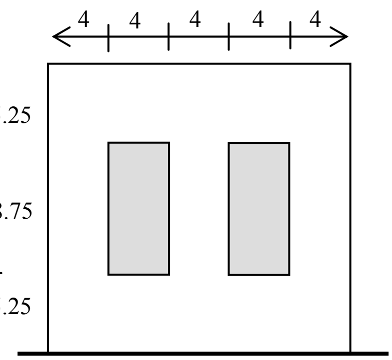


Figure 5c

Table 1 - Panel Figure 5a

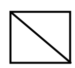
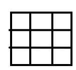
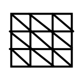
	Matrix A dimensions	LP α	NLP α	N°	mesh	Matrix A dimensions	$tg\varphi_d = tg\varphi_d^*$ $tg\varphi_v = k tg\varphi_o$	α
Ferris ... [1]	55 x 141	0.58	0.557	1a		6 x 6		0.57
Baggio... [2]	55 x 141	-----	0.545	1b		27 x 45	k=1 k=1.5 k=3	0.50 0.56 0.65
				1c		54 x 72	k=1 k=1.5 k=3	0.31 0.34 0.52

Table 2 - Panel Figure 5b

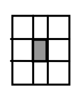
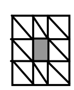
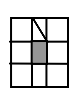
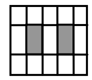
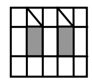
	Matrix A dimensions	LP α	NLP α	N°	mesh	Matrix A dimensions	$tg\varphi_d = tg\varphi_d^*$ $tg\varphi_v = k tg\varphi_o$	α
Ferris ... [1]	46 x 102	0.37	0.31	2a		24 x 33	k=1.5 k=3	0.35 0.45
Baggio... [2]	46 x 102	-----	0.35	2b		48 x 57	k=1.5 k=3	0.17 0.24
				2c		27 x 36	k=1.5 k=3	0.35 0.41

Table 3 - Panel Figure 5c

	Matrix A dimensions	LP α	NLP α	N°	mesh	Matrix A dimensions	$tg\varphi_d = tg\varphi_d^*$ $tg\varphi_v = ktg\varphi_o$	α
Ferris ... [1]	55 x 116	0.33	0.26	3a		39 x 57	k=3	0.27
Baggio... [2]	55 x 116	0.32	-----	3b		45 x 63	k=3	0.25

In Figures 6, 7 and 8, the collapse mechanisms of examples 1c (k=3), 2a (k=1,5) and 3b (k=3) are illustrated, being k an amplifier coefficient of $tg\varphi_o$.

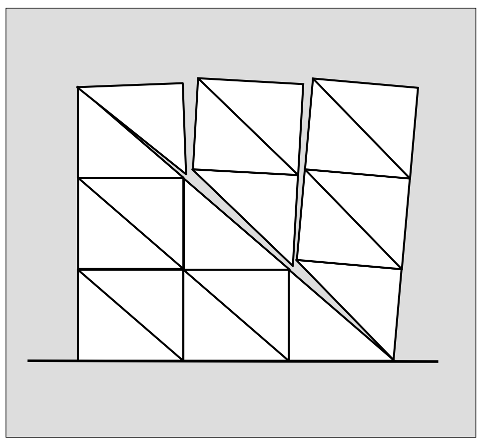


Figure 6 - Example 1c

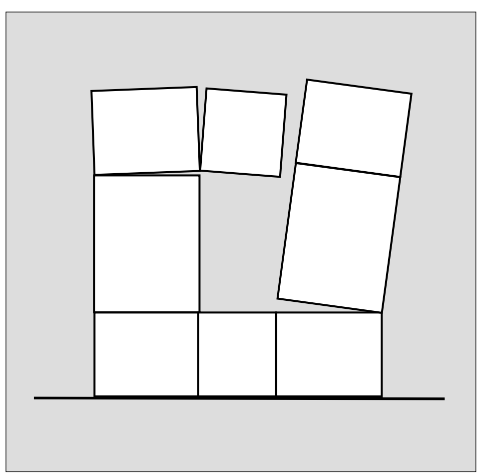


Figure 7 - Example 2a

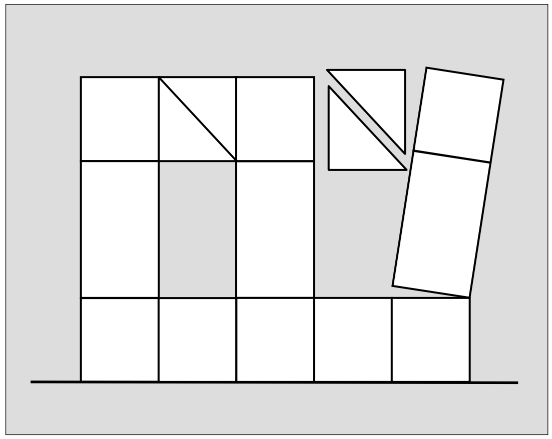


Figure 8 - Example 3b

CONCLUSIONS

Owing to the results obtained with the formulation proposed, we observe that in modelling masonry panels with rigid rectangular macroblocks, it is necessary to assign a value of the friction coefficient tgφ_v on the vertical interfaces higher than the value of the friction coefficient tgφ_o used for the horizontal interfaces, to take into account the modality of the texture of bricks or stones.

For walls without openings the introduction of diagonal lines, that is the modelling in rigid triangular macroblocks, leads to a considerable reduction in value of the multiplier α, even if we assign a value of the friction coefficient tgφ_d not lower than tgφ_d* on the diagonal interfaces. However, the above-mentioned reduction is very small if we only increase further the value of friction coefficient tgφ_v on the vertical interfaces.

Besides, for walls with openings the introduction of diagonal lines must be limited to the macroblocks above the openings only, as the examples into Tables 2 and 3 show.
Finally, the formulation proposed, is very simple because it involves few variables - the unknowns "hyperstatics" only. The formulation is suitable for determining the load collapse multiplier for masonry panels, as showed from the computational results that appear to be in good agreement with those achieved by other more sophisticated formulations.

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